# Math 255A Lecture 25 Notes

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## 1 The Weak$^*$ Topology and the Banach-Alaoglu Theorem

### 1.1 Completeness of the weak$^*$ topology

**Proposition 1.1.** *Let $\xi_n \in B^*$ be such that $\xi_n - \xi_m \to 0$ in $\sigma(B^*, B)$ as $n, m \to \infty$. Then there exists $\xi \in B^*$ such that $\xi_n \to \xi$ in $\sigma(B^*, B)$.*

*Proof.* We have $\langle x, \xi_m \rangle - \langle x, \xi_m \rangle \to 0$ for each $x \in B$, so the limit $\lim_{n\to\infty} \langle x, \xi_n \rangle$ exists pointwise, and we can let $\langle x, \xi \rangle = \lim_{n\to\infty}$ for $x \in B$. Then $\xi \in B^*$ by the Banach-Steinhaus theorem. $\square$

### 1.2 Tychonov's theorem and the Banach-Alaoglu theorem

**Theorem 1.1** (Banach-Alaoglu)**.** *Let $B$ be a Banach space. Then the closed unit ball $U = \{\xi \in B^* : \|\xi\| \leq 1\}$ is compact in $\sigma(B^*, B)$.*

The main point in the proof is Tychonov's theorem from point set topology. Let's review this.

Let $(X_\alpha)_{\alpha \in J}$ be a collection of topological spaces. Then the product space $X = \prod_{\alpha \in J} X_\alpha = \{f : J \to \bigcup_{\alpha \in J} X_\alpha \mid f(\alpha) \in X_\alpha \ \forall \alpha \in J\}$. is equipped with the product topology, the weakest topology such that the projection maps $p_\alpha : X \to X_\alpha$ sending $x \mapsto x_\alpha$ (where $x = \{x_\alpha\}_{\alpha \in J}$) are continuous for all $\alpha$. A base for the product topology is given by the finite intersection $\bigcap_{\text{finite}} p_\alpha^{-1}(O_\alpha)$, where $O_\alpha \subseteq X_\alpha$ is open.

**Theorem 1.2** (Tychonov)**.** *If $X_\alpha$ is compact for all $\alpha \in J$, then the space $X = \prod_{\alpha \in J} X_\alpha$ is compact in the product topology.*

We will not prove this, but we will use this in our proof of the Banach-Alaoglu theorem.

*Proof.* When $x \in B$, let $D_x = \{z \in K : |z| \leq \|x\|\}$. If $\xi \in U = \{\xi \in B^* : \|\xi\| \leq 1\}$, then $\langle x, \xi \rangle \in D_x$ for all $x$. Consider the injective map $\gamma : U \to D = \prod_{x \in B} D_x$ sending $\xi \mapsto \{\langle x, \xi \rangle\}_{x \in B}$. Equip $U$ with the weak$^*$ topology and $D$ with the product topology.

We claim that $\gamma$ is continuous. Let $O$ be an open set in $D$. We can assume that $O = \{f = (f_x)_{x\in B} : |f_{x_j} - cx_j| < \varepsilon_{x_j}, \varepsilon_{x_j} > 0, c_{x_j} \in D_{x_j}, 1 \leq j \leq N\}$. Then the inverse image $\gamma^{-1}(O) = \{\xi \in U : |\langle x_j, \xi\rangle - c_{x_j}| < \varepsilon_{x_j}, 1 \leq j \leq N\}$ is open in $\sigma(B^*, B)$. Similarly, $\gamma^{-1} : \mathrm{im}(\gamma) \to U$ is continuous. So $\gamma : U \to \mathrm{im}(\gamma)$ is a homeomorphism.

It suffices to check that $\mathrm{im}(\gamma) \subseteq D$ is compact in the product topology. By Tychonov's theorem, $D$ is compact, so we only need that $\mathrm{im}(\gamma)$ is closed. We have that

$$\begin{aligned}
\mathrm{im}(\gamma) &= \{f = (f_x)_{x\in B} \in D : f_{x+y} = f_x + f_y, f_{\lambda x} = \lambda f_x \;\forall x, y \in B, \forall \lambda \in \mathbb{C}\} \\
&= \bigcap_{x,y\in B} \{f : f_{x+y} = f_x + f_y\} \cap \bigcap_{\substack{\lambda\in\mathbb{C} \\ x\in B}} \{f : f_{\lambda_x} = \lambda f_x\}.
\end{aligned}$$

We now claim that $E_{x,\lambda} := \{f = (f_y)_{y\in B} : f_{\lambda x} = \lambda f_x\}$ is closed in $D$. Let $f_0 \in \overline{E_{x,\lambda}}$. An open neighborhood of $f_0$ is a set of the form $V_{x,\varepsilon} := \{f \in D : |f_x - f_{0,x}| < \varepsilon\}$. Let $f \in E_{x,\lambda} \cap V_{\lambda x,\varepsilon} \cap V_{x,\varepsilon}$; $V_{\lambda x,\varepsilon} \cap V_{x,\varepsilon}$ is an open neighborhood of $f_0$. Then

$$\begin{aligned}
|f_{0,\lambda x} - \lambda f_{0,x}| &= |f_{0,\lambda x} - f_{\lambda x} + \lambda f_x - \lambda f_{0,x}| \\
&\leq |f_{0,\lambda x} - f_{\lambda x}| + |\lambda||f_x - f_{0,x}| \\
&\leq \varepsilon + |\lambda|\varepsilon,
\end{aligned}$$

so $f_0 \in E_{x,\lambda}$. The result follows. □

Now that we have proved the theorem in full generality, it is worth noting that for separable Banach spaces, there is an elementary proof.

**Proposition 1.2.** *Let $B$ be a separable Banach space, and let $x_1, x_2, \ldots$ be a dense subset. Then the seminorms $\xi \mapsto |\langle x_k, \xi\rangle|$ for $j = 1, 2, \ldots$ define the same ropology as $\sigma(B^*, B)$.*

*Proof.* □

**Corollary 1.1.** *Let $B$ be a separable Banach space. Then $U = \{\xi \in B^* : \|\xi\| \leq 1\}$ is a compact metrizable space in the weak* topology $\sigma(B^*, B)$.*

*Proof.* If $\|\xi_n\| \leq 1$ for $n = 1, 2, \ldots$, then there exists a subsequence $(\xi_{n_k})$ converging in $\sigma(B^*, B)$. □